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Spacelike hypersurfaces in de Sitter space

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Abstract

In this paper, we give one intrinsic inequality for spacelike hypersurfaces in de Sitter space and a sufficient and necessary condition for such hypersurfaces to be totally geodesic. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $M_p^{n+1}(c)$ be an $(n+1)$-dimensional connected semi-Riemannian manifold of constant curvature c whose index is p. It is called an *indefinite space form of index p* and simply a *space form* when $p = 0$. If $c > 0$, we call $M_1^{n+1}(c)$ a *de Sitter space* of index 1. Let M^n be an n-dimensional Riemannian manifold immersed in $M_1^{n+1}(c)$. The semi-Riemannian metric of $M_1^{n+1}(c)$ induces the Riemannian metric of M^n, M^n is called a spacelike hypersurface. Montiel [1] gave an integral inequality for compact spacelike hypersurfaces in the de Sitter space and by use of this integral inequality, he studied the constant mean curvature spacelike hypersurface. There are other literatures studied spacelike hypersurfaces [2–7]. In this paper, we consider the general spacelike hypersurfaces in de Sitter space of index 1 and obain an intrinsic inequalitiy for such hypersurfaces. We also give a sufficient and necessary condition for such hypersurfaces to be totally geodesic. Then,we apply our theorem to Einstein spacelike hypersurface. In this paper, all the manifolds are C^∞ manifolds. We will prove the following.

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Theorem 1. *M^n is a spacelike hypersurface of de Sitter space $M_1^{n+1}(c)$ $(c>0)$, S and ρ are Ricci curvature tensor and the scalar curvature of M^n, respectively, then*

$$|S|^2 \geq 2c\rho(n-1) - c^2 n(n-1)^2.$$

Theorem 2. *M^n is a spacelike hypersurface of de Sitter space $M_1^{n+1}(c)$ $(c>0)$, S and ρ are Ricci curvature tensor and the scalar curvature of M^n, respectively, then $|S|^2 = 2c\rho(n-1) - c^2 n(n-1)^2$ if and only if M^n is totally geodesic.*

Corollary 1. *M^n is a spacelike Einstein hypersurface of de Sitter space $M_1^{n+1}(c)(c>0)$ with $\mathrm{Ric} = c(n-1)g$ (g is the Riemannian metric of M^n), then M^n is totally geodesic.*

2. Preliminary

We choose a local field of semi-Riemannian orthonormal frames $\{e_1, \ldots, e_n, e_{n+1}\}$ in $M_1^{n+1}(c)$ such that, restricted to M^n, $e_1, \ldots, e_n$ are tangent to M^n. Let $\omega_1, \ldots, \omega_{n+1}$ be its dual frame field such that the semi-Riemannian metric of $M^{n+1}(c)$ is given by $\mathrm{d}s^2 = \sum(\omega_i)^2 - (\omega_{n+1})^2 = \sum \epsilon_A(\omega_A)^2$, where $\epsilon_i = 1, i = 1, \ldots, n$ and $\epsilon_{n+1} = -1$. Then the structure equations of $M^{n+1}(c)$ are given by

$$d\omega_A = -\sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} + \tfrac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D, \tag{2.2}$$

$$K_{ABCD} = c\epsilon_A \epsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.3}$$

We restrict these forms to M^n,then

$$\omega_{n+1} = 0, \tag{2.4}$$

and the Riemannian metric of M^n is writen as $\mathrm{d}s^2 = \sum(\omega_i)^2$. Since $0 = \mathrm{d}\omega_{n+1} = -\sum \omega_{n+1,i} \wedge \omega_i$, by Cartan's lemma we may write

$$\omega_{n+1,i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.5}$$

From these formulas, we obtain the structure equations of M^n:

$$\mathrm{d}\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.6}$$

$$\mathrm{d}\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \tfrac{1}{2} \sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.7}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.8}$$

where R_{ijkl} are the components of the curvature tensor of M^n. We call $h = \sum h_{ij}\omega_i \otimes \omega_j$, the second fundamental form of M^n. The mean curvature H of M^n is defined by $H =$

$(1/n)\sum_i h_{ii}$. At any point $x_0 \in M^n$, for the symmetry of (h_{ij}), we can make $h_{ij} = \lambda_i\delta_{ij}$ choosing suitable orthonormal frames, then we have

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \lambda_i\lambda_j(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{2.9}$$

3. Proofs of the theorems

In order to prove the theorems, we need the following:

Lemma 1 (Cauchy–Swartz inequality). *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$, be real numbers, then*

$$\left(\sum_i a_i b_i\right)^2 \le \sum_i (a_i)^2 \sum_j (b_j)^2, \tag{3.1}$$

and the equality holds if and only if there exist a constant λ such that $a_i = \lambda b_i$, $i = 1, \ldots, n$, or $b_i = \lambda a_i$, $i = 1, \ldots, n$.

Lemma 2. *Let M^n is a spacelike hypersurface of $M_1^{n+1}(c)$ $(c > 0)$. If the second fundamental form of M^n at any point x can be writen as*

$$\begin{pmatrix} \lambda(x) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

by choosing suitable orthonormal frames, where $\lambda(x)$ is a function on M^n, then M^n is totally geodesic.

Proof of the Lemma 2. First, it is clearly that $\lambda(x) = nH(x)$, thus, $\lambda(x)$ is a C^∞ function on M^n. We let

$$\nabla\lambda(x) = \sum_i \lambda_i(x)\omega_i, \quad \nabla^2\lambda(x) = \sum_{i,j} \lambda_{ij}(x)\omega_i \otimes \omega_j, \quad \nabla h = \sum_{i,j,k} h_{ijk}\omega_i \otimes \omega_j \otimes \omega_k,$$

by hypothesis and Eq. (2.9), we have □

$$h_{11} = \lambda(x), h_{ij} = 0, \quad (i, j) \neq (1, 1). \tag{3.2}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{3.3}$$

$$\rho = \sum_{i,j} R_{ijij} = cn(n-1) \tag{3.4}$$

The Laplacian of $\sigma = \sum_{i,j}(h_{ij})^2$ is given by

$$\begin{aligned}\tfrac{1}{2}\Delta\sigma &= \sum_{i,j,k}(h_{ijk})^2 + \sum_{i,j} h_{ij}\Delta h_{ij}\\ &= \sum_{i,j,k}(h_{ijk})^2 + n\sum_{i,j} h_{ij}H_{ij} + \sum_{i,j,k,m} h_{ij}h_{im}R_{mkjk} + \sum_{i,j,k,m} h_{ij}h_{km}R_{mijk}.\end{aligned}$$

By Eqs. (3.2) and (3.3), we have

$$\tfrac{1}{2}\Delta\sigma = \sum_{i,j,k}(h_{ijk})^2 + \lambda(x)\sum_i \lambda_{ii}(x) + c(n-1)\lambda^2(x). \tag{3.5}$$

On the other hand, we have

$$\sigma = \lambda^2(x)$$

Thus,

$$\tfrac{1}{2}\Delta\sigma = \tfrac{1}{2}\Delta\lambda^2(x) = \sum_i (\lambda_i(x))^2 + \lambda(x)\sum_i \lambda_{ii}(x). \tag{3.6}$$

Thus, Eq. (3.5) can be writen as

$$\sum_i (\lambda_i(x))^2 = \sum_{i,j,k}(h_{ijk})^2 + c(n-1)\lambda^2(x). \tag{3.7}$$

By Eq. (3.3) and Bonnet–Myers theorem, we know that M^n is compact. So there is a point x where $\lambda(x)$ attains its maximum and at this point

$$\lambda_{ii}(x) \le 0, \quad \lambda_i(x) = 0 \tag{3.8}$$

By Eq. (3.7), this imply that $\max_{x\in M^n}\lambda(x) = 0$. In the same way, we can get $\min_{x\in M^n}\lambda(x) = 0$, hence, $\lambda(x) \equiv 0$, M^n is totally geodesic.

Proof of the Theorem 1. From Eq. (2.9),

$$\begin{aligned}S_{ij} &= \sum_k R_{kikj} = \sum_k \{-\lambda_k\lambda_i(\delta_{kk}\delta_{ij} - \delta_{kj}\delta_{ik}) + c(\delta_{kk}\delta_{ij} - \delta_{kj}\delta_{ik})\}\\ &= -\sum_k \lambda_k\lambda_i\delta_{ij} + \sum_k \lambda_i\lambda_j\delta_{kj}\delta_{ik} + c\sum_k(\delta_{ij} - \delta_{ki}\delta_{kj})\\ &= -\lambda_i\delta_{ij}\sum_k \lambda_k + \lambda_i\lambda_j\delta_{ij} + c(n-1)\delta_{ij}\end{aligned}$$

So

$$|S|^2 = \sum_{i,j} S_{ij}^2 = \sum_{i,j}\left\{-\lambda_i\delta_{ij}\sum_k \lambda_k + \lambda_i\lambda_j\delta_{ij} + c(n-1)\delta_{ij}\right\}^2$$

$$= \sum_{ij} \Big\{ \lambda_i^2 (\sum_k \lambda_k)^2 \delta_{ij} + \lambda_i^2 \lambda_j^2 \delta_{ij} + c^2 (n-1)^2 \delta_{ij} - 2\lambda_i^2 \lambda_j \delta_{ij} \sum_k \lambda_k - 2c(n-1)\lambda_i \delta_{ij} \sum_k \lambda_k + 2c(n-1)\lambda_i \lambda_j \delta_{ij} \Big\}$$

$$= \left(\sum_i \lambda_i^2\right)\left(\sum_k \lambda_k\right)^2 + \sum_i \lambda_i^4 + c^2 n(n-1)^2 - 2\sum_i \lambda_i^3 \sum_k \lambda_k - 2c(n-1)(\sum_i \lambda_i)^2 + 2c(n-1)\sum_i \lambda_i^2 \tag{3.9}$$

and

$$\rho = \sum_{i,j} R_{ijij} = \sum_{i,j} \{c(1-\delta_{ij}) - \lambda_i \lambda_j (1-\delta_{ij})\} = -\left(\sum_i \lambda_i\right)^2 + \sum_i \lambda_i^2 + cn(n-1) \tag{3.10}$$

So

$$\left(\sum_i \lambda_i\right)^2 = \sum \lambda_i^2 + cn(n-1) - \rho \tag{3.11}$$

$$|S|^2 = \left(\sum_i \lambda_i^2\right)\left(\sum_k \lambda_k\right)^2 + \sum_i \lambda_i^4 + c^2 n(n-1)^2 - 2\sum_i \lambda_i^3 \sum_k \lambda_k - 2c(n-1)\left(\sum_i \lambda_i\right)^2 + 2c(n-1)\sum_i \lambda_i^2$$

$$= \left(\sum_i \lambda_i^2\right)\left(\sum_i \lambda_i^2 + cn(n-1) - \rho\right) + \sum_i \lambda_i^4 + c^2 n(n-1)^2 - 2\sum_i \lambda_i^3 \sum_k \lambda_k - 2c(n-1)\left(\sum_i \lambda_i^2 + cn(n-1) - \rho\right)$$

$$+ 2c(n-1)\sum_i \lambda_i^2 \geq \left(\sum_i \lambda_i^2\right)\left(\sum_i \lambda_i^2 + cn(n-1) - \rho\right) + \sum_i \lambda_i^4 - 2\left(\sum_i \lambda_i^4\right)^{1/2}\left(\sum_i \lambda_i^2\right)^{1/2}\left(\sum_i \lambda_i^2 + cn(n-1) - \rho\right)^{1/2}$$

$$+ 2c(n-1)\rho - c^2 n(n-1)^2$$

$$= \left\{ \left(\sum_i \lambda_i^4 \right)^{1/2} - \left(\sum_i \lambda_i^2 \right)^{1/2} \left(\sum_i \lambda_i^2 + cn(n-1) - \rho \right)^{1/2} \right\}^2$$
$$+ 2c(n-1)\rho - c^2 n(n-1)^2 \geq 2c(n-1)\rho - c^2 n(n-1)^2 \tag{3.12}$$

the first inequality has used Lemma 1. □

Theorem 1 is proved.

Proof of the Theorem 2. If M^n is totally geodesic, i.e. $\lambda_i = 0, i = 1, \dots, n$, then from Eqs. (3.2) and (3.3),

$$|S|^2 = c^2 n(n-1)^2, \quad \rho = cn(n-1) \tag{3.13}$$

i.e. $|S|^2 = 2c\rho(n-1) - c^2 n(n-1)^2$.

Inversely, if Eq. (3.12) becomes an equality, then all the inequality of Eq. (3.12) will become equality. From Lemma 1, there exist a constant λ such that

$$\lambda_i^2 = \lambda \lambda_i, \quad i = 1, \dots, n \quad \text{or} \quad \lambda \lambda_i^2 = \lambda_i, \quad i = 1, \dots, n.$$

For simplicity, we assume $\lambda_i = \lambda, i = 1, \dots, k$ and $\lambda_j = 0, j = k+1, \dots, n$.

If $\lambda = 0$, then M^n is obviously totally geodesic.

Now, we assume $\lambda \neq 0$, so M^n is not totally geodesic. Because the second inequality of Eq. (3.12) should be equality, so

$$\left\{ \left(\sum_i \lambda_i^4 \right)^{1/2} - \left(\sum_i \lambda_i^2 \right)^{1/2} \left(\sum_i \lambda_i^2 + cn(n-1) - \rho \right)^{1/2} \right\}^2 = 0$$

i.e.

$$(k\lambda^4)^{1/2} - (k\lambda^2)^{1/2}(k\lambda^2 + cn(n-1) - \rho)^{1/2} = 0$$

Through simple calculating, we get

$$\rho = (k-1)\lambda^2 + cn(n-1) \tag{3.14}$$

On the other hand, by Eq. (3.3), we have

$$\rho = -\left(\sum_i \lambda_i \right)^2 + \sum_i \lambda_i^2 + cn(n-1) = (k - k^2)\lambda^2 + cn(n-1), \tag{3.15}$$

thus, $(k-1)\lambda^2 + cn(n-1) = (k-k^2)\lambda^2 + cn(n-1)$, so $k = 1$. By Lemma 2, M^n is totally geodesic. This is a contradiction. □

Proof of Corollary. In fact, if M^n is a spacelike Einstein hypersurface of $M_1^{n+1}(c)(c > 0)$ with Ric$= c(n-1)g$ (g is the Riemannian metric of M^n), then $\rho = cn(n-1)$ and $|S|^2 = c^2 n(n-1)^2 = 2c\rho(n-1) - c^2 n(n-1)^2$. The corollary follows immediately from Theorem 2. □

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